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On the waterbag model of dispersionless KP hierarchy

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Abstract
We investigate the bi-Hamiltonian structure of the waterbag model of dispersionless K (dKP) for the two-component case. One can establish the third- and first-order Hamiltonian operators associated with the waterbag model. Also, the dispersive corrections are discussed.

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1. Introduction

The dispersionless KP hierarchy (dKP or Benney moment chain) is defined by

$$\partial_{t_n}\lambda(p) = \{\lambda(p), B_n(p)\}, \qquad n = 1, 2, \ldots, \tag{1}$$

where the Lax operator $\lambda(p)$ is

$$\lambda(p) = p + \sum_1^{\infty} v_{n+1} p^{-n} \tag{2}$$

and it can be used to define a set of polynomials:

$$B_n(p) = \frac{[\lambda^n(p)]_+}{n}, \qquad i = 1, 2, 3, \ldots, \qquad t_1 = x.$$

Here $[\cdots]_+$ denotes a non-negative part of the Laurent series $\lambda^n(p)$. For example,

$$B_2 = \frac{p^2}{2} + v_2, \qquad B_3 = \frac{p^3}{3} + v_2 p + v_3.$$

Moreover, the bracket in (1) stands for the natural Poisson bracket on the space of functions of the two variables (x, p):

$$\{f(x, p), g(x, p)\} = \partial_x f \partial_p g - \partial_x g \partial_p f. \tag{3}$$

The compatibility of (1) will imply the zero-curvature equation

$$\partial_m B_n(p) - \partial_n B_m(p) = \{B_n(p), B_m(p)\}. \tag{4}$$

If we denote $t_2 = y$ and $t_3 = t$, then equation (4) for $m = 2, n = 3$ gives

$$v_{3x} = v_{2y}, \qquad v_{3y} = v_{2t} - v_2 v_{2x},$$

from which the dKP equation is derived ($v_2 = v$)

$$v_{yy} = (v_t - v v_x)_x. \tag{5}$$

According to the dKP theory [1, 18, 22, 37], there exists a wavefunction $S(\lambda, x, t_2, t_3, \ldots)$ such that $p = S_x$ and satisfies the Hamiltonian–Jacobian equation

$$\frac{\partial S}{\partial t_n} = B_n(p)|_{p=S_x}. \tag{6}$$

It can be seen that the compatibility of (6) also implies the zero-curvature equation (4). Now, we expand $B_n(p)$ as

$$B_n(p(\lambda)) = \frac{[\lambda^n(p)]_+}{n} = \frac{\lambda^n}{n} - \sum_{i=0}^{\infty} G_{in}\lambda^{-i-1},$$

where the coefficients can be calculated by the residue form,

$$G_{in} = -\mathrm{res}_{\lambda=\infty}(\lambda^i B_n(p)\,\mathrm{d}\lambda) = \frac{1}{i+1}\mathrm{res}_{p=\infty}\left(\lambda^{i+1}\frac{\partial B_n(p)}{\partial p}\,\mathrm{d}p\right),$$

which also shows the symmetry property

$$G_{in} = G_{ni}.$$

Moreover, from

$$\frac{\partial B_m(\lambda)}{\partial t_n} = \frac{\partial B_n(\lambda)}{\partial t_m}$$

this implies the integrability of G_{in} as expressed in terms of the free energy $\mathcal{F}$ (dispersionless τ function) [37]

$$G_{in} = \frac{\partial^2 \mathcal{F}}{\partial t_i \partial t_n}.$$

For example, the series inverse to (2) is

$$p = \lambda - \frac{\mathcal{F}_{11}}{\lambda} - \frac{\mathcal{F}_{12}}{2\lambda^2} - \frac{\mathcal{F}_{13}}{3\lambda^3} - \frac{\mathcal{F}_{14}}{4\lambda^4} - \cdots, \tag{7}$$

where $\mathcal{F}_{1n}$ are polynomials of $v_2, v_3, \ldots, v_{n+1}$ and in fact

$$h_n \equiv \frac{\mathcal{F}_{1n}}{n} = \mathrm{res}_{p=\infty}\frac{\lambda^n}{n}\,\mathrm{d}p$$

are the conserved densities for the dKP hierarchy (1). In [5, 6], it is proved that dKP hierarchy (1) is equivalent to the dispersionless Hirota equation

$$D(\lambda)S(\lambda') = -\log\frac{p(\lambda) - p(\lambda')}{\lambda}, \tag{8}$$

where $D(\lambda)$ is the operator $\sum_{n=1}^{\infty}\frac{1}{n\lambda^n}\frac{\partial}{\partial t_n}$.

Next, we consider the symmetry constraint [3, 4]

$$\mathcal{F}_x = \sum_{i=1}^{n} c_i(S_i - \tilde{S}_i), \tag{9}$$

where $S_i = S(\lambda_i)$ and $\tilde{S}_i = S(\tilde{\lambda}_i)$, $\lambda_i, \tilde{\lambda}_i$ are some sets of points, and c_i are arbitrary constants. Note that from (7) we know

$$D(\lambda)\mathcal{F}_x = \lambda - p.$$

On the other hand, by (8) and (9), we also have

$$D(\lambda)\mathcal{F}_x = D(\lambda)\sum_{i=1}^{N} c_i(S_i - \tilde{S}_i) = \sum_{i=1}^{N} c_i(D(\lambda)S_i - D(\lambda)\tilde{S}_i)$$
$$= -\sum_{i=1}^{N} c_i \log\frac{p-p_i}{p-\tilde{p}_i},$$

where $p = p(\lambda)$, $p_i = p(\lambda_i)$ and $\tilde{p}_i = p(\tilde{\lambda}_i)$.

Hence we obtain the non-algebraic reduction ('waterbag' model) [3, 16] of dKP hierarchy

$$\lambda = p - \sum_{i=1}^{N} c_i \log\frac{p-p_i}{p-\tilde{p}_i} = p + \sum_{s=1}^{\infty} v_{s+1} p^{-s}, \tag{10}$$

where

$$v_{s+1} = \sum_{i=1}^{N} \frac{c_i\left(p_i^s - \tilde{p}_i^s\right)}{s}. \tag{11}$$

One remarks that in the limit $\tilde{\lambda}_i = \lambda_i + \epsilon_i$, $\epsilon_i \to 0$, keeping $c_i\epsilon_i = d_i$ to be constant, the Sato function (10) reproduces the Zakharov's reduction [38]

$$\lambda = p + \frac{d_1}{p-\tilde{p}_1} + \frac{d_2}{p-\tilde{p}_2} + \cdots + \frac{d_N}{p-\tilde{p}_N}, \tag{12}$$

which is the algebraic reduction of dKP hierarchy.

From (10), we have

$$B_2(p) = \frac{1}{2}p^2 + \sum_{i=1}^{N} c_i(p_i - \tilde{p}_i).$$

So ($t_2 = y$)

$$\partial_y \lambda = \left\{\lambda, \sum_{i=1}^{N} c_i(p_i - \tilde{p}_i)\right\}.$$

This will imply

$$\partial_y p_\lambda = \partial_x\left[\frac{1}{2}p_\lambda^2 + \sum_{i=1}^{N} c_i(p_i - \tilde{p}_i)\right] \qquad \partial_y \tilde{p}_\lambda = \partial_x\left[\frac{1}{2}\tilde{p}_\lambda^2 + \sum_{i=1}^{N} c_i(p_i - \tilde{p}_i)\right]. \tag{13}$$

For simplicity, in this paper we only consider the case $N = 1$, i.e.,

$$\partial_y p_1 = \partial_x\left[\tfrac{1}{2}p_1^2 + c_1(p_1 - \tilde{p}_1)\right] \qquad \partial_y \tilde{p}_1 = \partial_x\left[\tfrac{1}{2}\tilde{p}_1^2 + c_1(p_1 - \tilde{p}_1)\right], \tag{14}$$

and the Lax operator (10) is truncated to become

$$\lambda = p - c_1 \log\frac{p-p_1}{p-\tilde{p}_1}. \tag{15}$$

Equation (14) can also be written as the Hamiltonian system

$$\begin{bmatrix} p_{1y} \\ \tilde{p}_{1y} \end{bmatrix} = \begin{bmatrix} \frac{1}{c_1} & 0 \\ 0 & -\frac{1}{c_1} \end{bmatrix} \partial_x \begin{bmatrix} \frac{\delta H_3}{\delta p_1} \\ \frac{\delta H_3}{\delta \tilde{p}_1} \end{bmatrix},$$

where δ is the variation derivative and

$$H_3 = \frac{1}{3}\int \mathrm{d}x\left[c_1\left(p_1^3 - \tilde{p}_1^3\right) + 3c_1^2(p_1 - \tilde{p}_1)^2\right].$$

A bi-Hamiltonian structure is defined as (for the case of dKP)

$$\begin{bmatrix} p_{1y} \\ \tilde{p}_{1y} \end{bmatrix} = \mathbb{J}_1 \begin{bmatrix} \frac{\delta H_3}{\delta p_1} \\ \frac{\delta H_3}{\delta \tilde{p}_1} \end{bmatrix} = \mathbb{J}_2 \begin{bmatrix} \frac{\delta H}{\delta p_1} \\ \frac{\delta H}{\delta \tilde{p}_1} \end{bmatrix},$$

where

$$\mathbb{J}_1 = \begin{bmatrix} \frac{1}{c_1} & 0 \\ 0 & -\frac{1}{c_1} \end{bmatrix} \partial_x$$

and H is some Hamiltonian such that $\mathbb{J}_2$ is also a Hamiltonian operator (Jacobi identity), which is compatible with $\mathbb{J}_1$, i.e., $\mathbb{J}_1 + c\mathbb{J}_2$ is also a Hamiltonian one for any complex number c [12, 13, 23, 29]. We hope to find $\mathbb{J}_2$ and the related Hamiltonian H.

Besides, from (11) and (15) we know that $v_2 = v = p_1 - \tilde{p}_1$. Then by the theory of symmetry constraints of KP [8, 20, 21, 35] hierarchy one can consider the dispersive corrections for the waterbag model (15).

This paper is organized as follows. In the next section, we construct the third-order bi-Hamiltonian structure. Section 3 is devoted to establishing first-order bi-Hamiltonian structure using WDVV equation in topological field theory. In section 4, we discuss the dispersive corrections. In the final section, one discusses some problems to be investigated.

2. Third-order bi-Hamiltonian structure

In this section, we investigate the bi-Hamiltonian structure of the two-component case (14). To find the bi-Hamiltonian structure, one can introduce the coordinates

$$u = p_1 + \tilde{p}_1, \qquad v = p_1 - \tilde{p}_1$$

to rewrite equation (14) as

$$\begin{pmatrix} u \\ v \end{pmatrix}_y = \begin{pmatrix} \frac{u^2+v^2}{4} + 2c_1 v \\ \frac{uv}{2} \end{pmatrix}_x = \frac{1}{c_1} \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H_3}{\delta u} \\ \frac{\delta H_3}{\delta v} \end{pmatrix}, \tag{16}$$

where

$$H_3 = \frac{1}{3} \int \mathrm{d}x \left[c_1 \left(p_1^3 - \tilde{p}_1^3 \right) + 3c_1^2 (p_1 - \tilde{p}_1)^2 \right] = \frac{1}{3} \int \mathrm{d}x \left[c_1 \frac{3u^2 v + v^3}{4} + 3c_1^2 v^2 \right] = \int \mathrm{d}x h_3$$

and δ is the variational derivative. One can observe that the conserved density

$$h_3 = \frac{1}{3} \left[c_1 \frac{3u^2 v + v^3}{4} + 3c_1^2 v^2 \right]$$

has the separable property

$$\frac{\partial^2 h_3}{\partial u^2} \bigg/ \frac{\partial^2 h_3}{\partial v^2} = \frac{c_1 v}{2} \bigg/ \left(\frac{c_1 v}{2} + 2c_1^2 \right) = \frac{1}{\mu(v)},$$

where

$$\mu(v) = \frac{v + 4c_1}{v} = 1 + \frac{4c_1}{v}.$$

Hence we can identify equation (16) as the generalized gas dynamic Hamiltonian system or separable Hamiltonian system [32, 34]. Therefore, according to the separable Hamiltonian theory in [2, 32], we know that the third-order Hamiltonian operator

$$J_2 = D_x U_x^{-1} D_x U_x^{-1} \sigma D_x,$$

where

$$U_x = \begin{pmatrix} u_x & \mu(v) v_x \\ v_x & u_x \end{pmatrix} \qquad \text{and} \qquad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

or

$$U_x^{-1} = \frac{1}{u_x^2 - \mu(v) v_x^2} \begin{pmatrix} u_x & -\mu(v) v_x \\ -v_x & u_x \end{pmatrix},$$

is compatible with the first-order Hamiltonian operator

$$J_1 = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix} = \sigma D_x.$$

So we can write equation (16) as the bi-Hamiltonian structure

$$\begin{pmatrix} u \\ v \end{pmatrix}_y = \frac{1}{c_1} J_1 \begin{pmatrix} \frac{\delta H_3}{\delta u} \\ \frac{\delta H_3}{\delta v} \end{pmatrix} = \frac{1}{c_1} J_2 \begin{pmatrix} \frac{\delta H_5}{3\delta u} \\ \frac{\delta H_5}{3\delta v} \end{pmatrix}$$
$$= \frac{1}{c_1} D_x \begin{pmatrix} \frac{h_{5,uuv}}{3} \\ \frac{h_{5,vvu}}{3} \end{pmatrix},$$

where

$$H_5 = \int h_5 \, \mathrm{d}x = \frac{1}{5} \int \mathrm{res}_{p=\infty} (\lambda^5 \, \mathrm{d}p) \, \mathrm{d}x$$
$$= \frac{1}{5} \int \left\{ c_1 \left(p_1^5 - \tilde{p}_1^5\right) + 10 c_1^3 (p_1 - \tilde{p}_1)^3 + \frac{20}{3} c_1^2 (p_1 - \tilde{p}_1)\left(p_1^3 - \tilde{p}_1^3\right) + \frac{5}{2} c_1^2 \left(p_1^2 - \tilde{p}_1^2\right)^2 \right\} \mathrm{d}x$$
$$= \frac{1}{5} \int \left\{ \frac{c_1}{16} (5u^4 v + 10 u^2 v^3 + v^5) + 10 c_1^3 v^3 + \frac{15}{2} c_1^2 u^2 v^2 + \frac{5}{3} c_1^2 v^4 \right\} \mathrm{d}x.$$

Next, we will find all the conserved density $F(u, v)$ of equation (16). It is not difficult to see that $\int F(u, v) \, \mathrm{d}x$ is a conserved quantity if and only if $F(u, v)$ satisfies the wave equation

$$F_{uu} = \frac{F_{vv}}{\mu(v)}. \tag{17}$$

The wave equation (17) leads to two fundamental hierarchies of conserved densities [32, 34]

$$F_N = \sum_{n=0}^{[\frac{N}{2}]} \frac{u^{N-2n}}{(N-2n)!} \left(\partial_v^{-2} \mu(v)\right)^n \cdot v \qquad \tilde{F}_N = \sum_{n=0}^{[\frac{N+1}{2}]} \frac{u^{N+1-2n}}{(N+1-2n)!} \left(\partial_v^{-2} \mu(v)\right)^n \cdot 1. \tag{18}$$

Here $\partial_v^{-1} = \int_0^v \mathrm{d}v$ and ∂_v^{-1} acts on all the factors standing to the right of it. For example,

$$\left(\partial_v^{-2} \mu(v)\right)^2 \cdot v = \int_0^v \mathrm{d}v \int_0^v \left[\mu(v) \int_0^v \mathrm{d}v \int_0^v v \mu(v) \, \mathrm{d}v \right] \mathrm{d}v.$$

For reference, we list the first few members of each sequence in the generalized gas dynamic system (16): $\left(\mu(v) = 1 + \frac{4c_1}{v}\right)$

$$F_0 = v, \qquad F_1 = uv$$
$$F_2 = \frac{1}{2}u^2 v + \frac{1}{6}v^3 + 2c_1 v^2 \qquad F_3 = \frac{1}{6}u^3 v + u\left(\frac{1}{6}v^3 + 2c_1 v^2\right)$$
$$F_4 = \frac{1}{24}u^4 v + \frac{u^2}{2}\left(\frac{1}{6}v^3 + 2c_1 v^2\right) + \left(\frac{1}{120}v^5 + \frac{2}{9}c_1 v^4 + \frac{4}{3}c_1^2 v^3\right)$$

and

$$\tilde{F}_0 = u, \qquad \tilde{F}_1 = \frac{1}{2}u^2 + \frac{1}{2}v^2 + 4c_1(v\ln v - v)$$
$$\tilde{F}_2 = \frac{1}{6}u^3 + u\left[\frac{1}{2}v^2 + 4c_1(v\ln v - v)\right]$$
$$\tilde{F}_3 = \frac{1}{24}u^4 + \frac{1}{2}u^2\left[\frac{1}{2}v^2 + 4c_1(v\ln v - v)\right] + \frac{1}{24}v^4 + \frac{2}{3}c_1 v^3 \ln v - \frac{8}{9}c_1 v^3 + 8c_1^2 v^2 \ln v - 20c_1^2 v^2$$
$$\tilde{F}_4 = \frac{1}{120}u^5 + \frac{1}{6}u^3\left[\frac{1}{2}v^2 + 4c_1(v\ln v - v)\right] + u\left[\frac{1}{24}v^4 + \frac{2}{3}c_1 v^3 \ln v - \frac{8}{9}c_1 v^3 + 8c_1^2 v^2 \ln v - 20c_1^2 v^2\right].$$

In fact, one can see that

$$F_{N-1} = \frac{2^{N-1}}{c_1(N-1)!}h_N = \frac{2^{N-1}}{c_1 N!}\operatorname{res}_{p=\infty}(\lambda^N\,\mathrm{d}p), \qquad N \geqslant 1 \tag{19}$$

and from (18) we also have (for $N \geqslant 1$)

$$\frac{\partial F_N}{\partial u} = F_{N-1}, \qquad \frac{\partial \tilde{F}_N}{\partial u} = \tilde{F}_{N-1} \tag{20}$$

Moreover, we note that the recursion operator

$$\hat{R} = J_2 J_1^{-1} = D_x U_x^{-1} D_x U_x^{-1} \tag{21}$$

is the square of a simpler first-order recursion operator

$$R = D_x U_x^{-1}.$$

Then we can easily check that, using (17) and (20),

$$R^{-1}\sigma D\begin{pmatrix}\frac{\partial F_N}{\partial u}\\ \frac{\partial F_N}{\partial v}\end{pmatrix} = R^{-1}\begin{pmatrix}\frac{\partial F_N}{\partial v}\\ \frac{\partial F_N}{\partial u}\end{pmatrix}_x = U_x\begin{pmatrix}\frac{\partial F_N}{\partial v}\\ \frac{\partial F_N}{\partial u}\end{pmatrix}$$
$$= \begin{pmatrix}\frac{\partial F_{N+1}}{\partial v}\\ \frac{\partial F_{N+1}}{\partial u}\end{pmatrix}_x = \sigma D\begin{pmatrix}\frac{\partial F_{N+1}}{\partial u}\\ \frac{\partial F_{N+1}}{\partial v}\end{pmatrix},$$

and similarly for $\tilde{F}_N$. Hence the dKP hierarchy of (16) can be obtained using the recursion operator, i.e.,

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_n} = (R^{-1})^{n-2} \frac{\sigma}{c_1} D \begin{pmatrix} \frac{\partial h_3}{\partial u} \\ \frac{\partial h_3}{\partial v} \end{pmatrix}, \qquad n \geqslant 2.$$

However, the Lax representation of hierarchy generated by $\tilde{F}_N$ is not found.

Also, using the recursion operator (21), one can construct a hierarchy of higher order Hamiltonian densities $\hat{F}_m, m = 1, 2, 3, \ldots,$ with m indicating the order of derivatives on which they depend, and the corresponding commuting bi-Hamiltonian system [31]:

$$\begin{pmatrix} u \\ v \end{pmatrix}_{\tau_m} = \hat{R}^m \begin{pmatrix} 1 \\ 0 \end{pmatrix} = J_1 \begin{pmatrix} \frac{\delta \hat{F}_m}{\delta u} \\ \frac{\delta \hat{F}_m}{\delta v} \end{pmatrix} = J_2 \begin{pmatrix} \frac{\delta \hat{F}_{m-1}}{\delta u} \\ \frac{\delta \hat{F}_{m-1}}{\delta v} \end{pmatrix}, \tag{22}$$

where

$$\hat{F}_0 = \int x v \, \mathrm{d}x$$

$$\hat{F}_1 = -\frac{1}{2} \int \frac{v_x}{u_x^2 - \mu(v) v_x^2} \, \mathrm{d}x = -\frac{1}{2} \int \frac{v_x}{u_x^2 - \left(1 + \frac{4c_1}{v}\right) v_x^2} \, \mathrm{d}x.$$

All the flows of (22) will commute with the generalized gas dynamic system (16). For example, for $n = 1$, we have, after a simple calculation,

$$\begin{pmatrix} u \\ v \end{pmatrix}_{\tau_1} = \hat{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\mu^2 v_x^3 v_{xx} + 3\mu u_x^2 v_x v_{xx} + \mu\mu' v_x^5 + \mu' u_x^2 v_x^3 - u_x^3 u_{xx} - 3\mu v_x^2 u_x u_{xx}}{\left(u_x^2 - \mu v_x^2\right)^3} \\ \frac{\mu v_x^3 u_{xx} + 3 u_x^2 v_x u_{xx} - v_{xx} u_x^3 - 3\mu v_x^2 u_x v_{xx} - 2\mu' u_x v_x^4}{\left(u_x^2 - \mu v_x^2\right)^3} \end{pmatrix}_x,$$

where

$$\mu' = \frac{\mathrm{d}\mu}{\mathrm{d}v} = -\frac{4c_1}{v^2}.$$

Finally, one remarks that there exists a Lagrangian local in the velocity fields for equation (16) (up to a scaling):

$$\mathcal{L} = \frac{v_x u_t - u_x v_t}{u_x^2 - \mu v_x^2} - 2v.$$

The local Lagrangian will exist in bi-Hamiltonian structure with a pair of first- and third-order Hamiltonian operators [30, 32].

3. Free energy and bi-Hamiltonian structure

In this section, we investigate the relations between bi-Hamiltonian structure and free energy. Then the compatible first-order Hamiltonian operators can be constructed.

Now, we want to find the free energy associated with the dKP hierarchy (1) for the Lax operator of the form (15). Suppose we are given two first-order Hamiltonian operators $\hat{J}_1$ and $\hat{J}_2$ $(\partial = \partial_x)$

$$\hat{J}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial \stackrel{\text{def}}{=} \eta_1^{ij} \partial$$

$$\hat{J}_2 = \begin{pmatrix} g^{11}(t) & g^{12}(t) \\ g^{21}(t) & g^{22}(t) \end{pmatrix} \partial + \begin{pmatrix} \Gamma_1^{11}(t) & \Gamma_1^{12}(t) \\ \Gamma_1^{21}(t) & \Gamma_1^{22}(t) \end{pmatrix} t_x^1 + \begin{pmatrix} \Gamma_2^{11}(t) & \Gamma_2^{12}(t) \\ \Gamma_2^{21}(t) & \Gamma_2^{22}(t) \end{pmatrix} t_x^2$$

$$\stackrel{\text{def}}{=} g^{ij}(t) \partial + \Gamma_k^{ij}(t) t_x^k.$$

They are both Poisson brackets of hydrodynamic type introduced by Dubrovin and Novikov [10, 11]. The bi-Hamiltonian structure means that $\hat{J}_1$ and $\hat{J}_2$ have to be compatible, i.e., $\hat{J} = \hat{J}_1 + \alpha\hat{J}_2$ must also be a Hamiltonian structure for all values of α. This compatibility condition implies that, for any α, the metric is referred to as flat pencil. The geometric setting in which to understand flat pencil (or the bi-Hamiltonian structure of a hydrodynamic system) is the Frobenius manifold [12–15]. One way to define such manifolds is to construct a function $\mathbb{F}(t^1, t^2, \ldots, t^m)$ such that the associated functions,

$$c_{ijk} = \frac{\partial^3\mathbb{F}}{\partial t^i\partial t^j\partial t^k},$$

satisfy the following conditions.

- The matrix $\eta_{ij} = c_{1ij}$ is constant and non-degenerate. This together with the inverse matrix η^{ij} are used to raise and lower indices. On such a manifold one may interpret η_{ij} as a flat metric.
- The functions $c^i_{jk} = \eta^{ir}c_{rjk}$ define an associative commutative algebra with a unity element. This defines a Frobenius algebra on each tangent space $T^t\mathcal{M}$.

Equations of associativity give a system of nonlinear PDE for $\mathbb{F}(t)$

$$\frac{\partial^3\mathbb{F}(t)}{\partial t^\alpha\partial t^\beta\partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3\mathbb{F}(t)}{\partial t^\mu\partial t^\gamma\partial t^\sigma} = \frac{\partial^3\mathbb{F}(t)}{\partial t^\alpha\partial t^\gamma\partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3\mathbb{F}(t)}{\partial t^\mu\partial t^\beta\partial t^\sigma}.$$

These equations constitute the Witten–Dijkgraaf–Verlinde–Verlinde (or WDVV) equations. On such a manifold one may introduce a second flat metric defined by

$$g^{ij} = \partial^i\partial^j\mathbb{F} + \partial^j\partial^i\mathbb{F}, \tag{23}$$

where

$$\partial^i = \eta^{i\varsigma}\partial_{t^\varsigma}$$

and the contravariant Levi-Civita connection is

$$\Gamma^{ij}_k = \partial^i\partial^j\partial_{t^k}\mathbb{F}. \tag{24}$$

This metric, together with the original metric η^{ij}, define a flat pencil (i.e, $\eta^{ij} + \alpha g^{ij}$ is flat for any value of α). Thus, one automatically obtains a bi-Hamiltonian structure from a Frobenius manifold $\mathcal{M}$. The corresponding Hamiltonian densities are defined recursively by the formula

$$\frac{\partial^2\psi^{(n)}_\alpha}{\partial t^i\partial t^j} = c^k_{ij}\frac{\partial\psi^{(n-1)}_\alpha}{\partial t^k}, \tag{25}$$

where $n \geqslant 1, \alpha = 1, 2, \ldots, m$, and $\psi^0_\alpha = \eta_{\alpha\epsilon}t^\epsilon$. The integrability conditions for this systems are automatically satisfied when the c^k_{ij} are defined as above.

For the waterbag hierarchy (1) and (15), it is obvious that

$$t^1 = u = \psi^{(o)}_2, \qquad t^2 = v = \psi^{(0)}_1$$

and those c^k_{ij} can be determined by (25)

$$\frac{\partial^2\psi^{(n)}_1}{\partial t^i\partial t^j} = c^k_{ij}\frac{\partial\psi^{(n-1)}_1}{\partial t^k},$$

where, using (19),

$$\psi^{(n)}_1 = F_n = \frac{2^n}{c_1(n+1)!}\operatorname{res}_{p=\infty}(\lambda^{n+1}\,\mathrm{d}p), \qquad n \geqslant 0.$$

Simple calculations can get

$$\begin{aligned} &c^1_{11} = 1, \qquad c^1_{12} = c^1_{21} = 0, \qquad c^1_{22} = 1 + \frac{4c_1}{t^2} = \mu(v), \\ &c^2_{11} = c^2_{22} = 0, \qquad c^2_{21} = c^2_{12} = 1. \end{aligned} \tag{26}$$

By (26), we can get immediately free energy

$$\mathbb{F}(t^1, t^2) = \tfrac{1}{2}(t^1)^2 t^2 + 2c_1 (t^2)^2 \log t^2 + \tfrac{1}{6}(t^2)^3 + \text{quadratic terms}. \tag{27}$$

We note that the free energy (27) has no quasi-homogeneity condition and, however, the free energy associated with the Benney hierarchy is quasi-homogeneous [7].

After choosing suitable quadratic terms, then from the free energy (27), using (23) and (24), one can construct $\hat{J}_2$ as follows:

$$\begin{aligned} \hat{J}_2 &= \begin{pmatrix} 2t^2 & 2t^1 \\ 2t^1 & 2t^2 \end{pmatrix} \partial + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t^1_x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t^2_x \\ &= \begin{pmatrix} t^2\partial_x + \partial_x t^2 & t^1\partial_x + \partial_x t^1 \\ t^1\partial_x + \partial_x t^1 & t^2\partial_x + \partial_x t^2 \end{pmatrix}. \end{aligned}$$

Here we remove the non-analytic part of g^{ij}, i.e., $\ln t^2$. Also, one can verify directly that $\hat{J}_2$ is a Hamiltonian operator and is compatible with $\hat{J}_1$. We note that the constant c_1 will not appear in $\hat{J}_2$.

Now, using the recursion operator

$$\hat{\mathbb{R}} = \hat{J}_2 \hat{J}_1^{-1} = \begin{pmatrix} t^1 + \partial_x t^1 \partial_x^{-1} & t^2 + \partial_x t^2 \partial_x^{-1} \\ t^2 + \partial_x t^2 \partial_x^{-1} & t^1 + \partial_x t^1 \partial_x^{-1} \end{pmatrix},$$

one can construct a hierarchy by

$$\begin{pmatrix} t^1 \\ t^2 \end{pmatrix}_{\tilde{\tau}_m} = \hat{\mathbb{R}}^m \begin{pmatrix} \frac{t^1}{6} \\ \frac{t^2}{6} \end{pmatrix}_x, \qquad m \geqslant 1. \tag{28}$$

For example, for $m = 1$ and $m = 2$, a simple calculation can yield

$$\begin{aligned} \begin{pmatrix} t^1 \\ t^2 \end{pmatrix}_{\tilde{\tau}_1} &= \hat{\mathbb{R}} \begin{pmatrix} \frac{t^1}{6} \\ \frac{t^2}{6} \end{pmatrix}_x = \begin{pmatrix} \frac{1}{4}[(t^1)^2 + (t^2)^2] \\ \frac{1}{2}t^1 t^2 \end{pmatrix}_x, \\ \begin{pmatrix} t^1 \\ t^2 \end{pmatrix}_{\tilde{\tau}_2} &= \hat{\mathbb{R}}^2 \begin{pmatrix} \frac{t^1}{6} \\ \frac{t^2}{6} \end{pmatrix}_x = 5 \begin{pmatrix} \frac{1}{12}(t^1)^3 + \frac{1}{4}t^1 (t^2)^2 \\ \frac{1}{12}(t^2)^3 + \frac{1}{4}t^2 (t^1)^2 \end{pmatrix}_x, \end{aligned}$$

which are slightly different from y flow (16) and t_3 (or t) flow of the dKP hierarchy (1), respectively:

$$\begin{pmatrix} t^1 \\ t^2 \end{pmatrix}_t = \frac{2}{3c_1} J_1 \begin{pmatrix} \frac{\delta H_4}{\delta t^1} \\ \frac{\delta H_4}{\delta t^2} \end{pmatrix} = \begin{pmatrix} \frac{1}{12}(t^1)^3 + \frac{1}{4}t^1(t^2)^2 + 2c_1 t^1 t^2 \\ \frac{1}{12}(t^2)^3 + \frac{1}{4}t^2(t^1)^2 + c_1 (t^2)^2 \end{pmatrix}_x, \tag{29}$$

where

$$H_4 = \int h_4 \,\mathrm{d}x = \frac{1}{4}\int \left\{ c_1 \left[\frac{(t^1)^3 t^2 + (t^2)^3 t^1}{2} \right] + 6c_1^2 t^1 (t^2)^2 \right\} \mathrm{d}x.$$

By comparisons between them, one can see that the non-homogeneous terms (or higher-order c_1 terms) of the waterbag hierarchy could be removed in the hierarchy (28). In this way, one can say that the hierarchy (28) is perturbed, up to some scalings, by the waterbag hierarchy with a perturbation parameter c_1.

Remark. According to the Kodama–Gibbons formulation [18], the Riemann invariants λ_1, λ_2 of (29) are given by

$$\begin{aligned}\lambda_1 = \lambda(u_1) &= \frac{t^1 + \sqrt{(t^2)^2 + 4c_1 t^2}}{2} - c_1 \ln \frac{\sqrt{(t^2)^2 + 4c_1 t^2} - t^2}{\sqrt{(t^2)^2 + 4c_1 t^2} + t^2} \\ \lambda_2 = \lambda(u_2) &= \frac{t^1 - \sqrt{(t^2)^2 + 4c_1 t^2}}{2} + c_1 \ln \frac{\sqrt{(t^2)^2 + 4c_1 t^2} - t^2}{\sqrt{(t^2)^2 + 4c_1 t^2} + t^2},\end{aligned} \tag{30}$$

where u_1 and u_2 are the real roots of $\frac{d\lambda}{dp}|_{p=u_1,u_2} = 0$, i.e.,

$$u_1 = \frac{t^1 + \sqrt{(t^2)^2 + 4c_1 t^2}}{2}, \qquad u_2 = \frac{t^1 - \sqrt{(t^2)^2 + 4c_1 t^2}}{2},$$

and the characteristic speeds are

$$\hat{v}_1 = \frac{\mathrm{d}\Omega_3(p)}{\mathrm{d}p}\bigg|_{p=u_1}, \qquad \hat{v}_2 = \frac{\mathrm{d}\Omega_3(p)}{\mathrm{d}p}\bigg|_{p=u_2}.$$

Then equation (29) can be written as

$$\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}_t = \begin{pmatrix}\hat{v}_1 & 0\\ 0 & \hat{v}_2\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}_x. \tag{31}$$

Also, a simple calculation shows that, using (31), the flat metric $(\mathrm{d}s)^2 = \mathrm{d}t^1\,\mathrm{d}t^2$ becomes, in Riemann's invariants,

$$(\mathrm{d}s)^2 = \eta_{11}(t)(\mathrm{d}\lambda_1)^2 + \eta_{22}(t)(\mathrm{d}\lambda_2)^2,$$

where

$$\begin{aligned}\eta_{11}(t) &= \mathrm{res}_{p_1}\frac{(\mathrm{d}p)^2}{\mathrm{d}\lambda} = \frac{1}{\frac{\mathrm{d}^2\lambda}{\mathrm{d}p^2}\big|_{p=u_1}} = \frac{t^2}{\sqrt{(t^2)^2 + 4t^2 c_1}} \\ \eta_{22}(t) &= \mathrm{res}_{p_2}\frac{(\mathrm{d}p)^2}{\mathrm{d}\lambda} = \frac{1}{\frac{\mathrm{d}^2\lambda}{\mathrm{d}p^2}\big|_{p=u_2}} = -\frac{t^2}{\sqrt{(t^2)^2 + 4t^2 c_1}}.\end{aligned}$$

Since it is known that waterbag reduction (15) is not scaling invariant [16], we can verify that the metric

$$\begin{aligned}(\mathrm{d}\tilde{s})^2 &= \frac{\eta_{11}}{u_1}(\mathrm{d}\lambda_1)^2 + \frac{\eta_{22}}{u_2}(\mathrm{d}\lambda_2)^2 \\ &= \frac{t^2}{4}(\mathrm{d}t^1)^2 + \frac{t^1}{2}\,\mathrm{d}t^1\,\mathrm{d}t^2 + \left(\frac{t^2}{4} + c_1\right)(\mathrm{d}t^2)^2\end{aligned} \tag{32}$$

is no more flat [15]. Hence from the theory of Darboux–Egrov metric [12], one believes that there is no first-order bi-Hamiltonian structure for (29)(or (16)). However, we know that (31) is (semi-)Hamiltonian [16, 33] and it probably will have a compatible non-local Poisson brackets of hydrodynamic type [24–28], deserving further investigations.

Finally, one notes that the metric (32) can also be obtained using the free energy (27)

$$\mathbb{F}(t^1, t^2) = \tfrac{1}{2}(t^1)^2 t^2 + 2c_1(t^2)^2 \log t^2 + \tfrac{1}{6}(t^2)^3 + 2c_1(t^1)^2.$$

4. Dispersive corrections

In this section, one will investigate the dispersive corrections of the waterbag model from the theory of symmetry constraints of KP hierarchy. Simple calculations show that the special symmetry constraint corresponding to the waterbag model is not admissible.

Let us briefly describe the KP hierarchy [9]. The Lax operator of KP hierarchy is

$$L = \partial_X + \sum_{n=1}^{\infty} V_{n+1} \partial_X^{-n},$$

and the KP hierarchy is determined by the Lax equation $\left(\partial_n = \frac{\partial}{\partial T_n}, T_1 = X\right)$

$$\partial_n L = [B_n, L],$$

where $B_n = \frac{1}{n} L_+^n$ is the differential part of L^n. For example $(T_2 = Y, T_3 = T)$

$$V_{2Y} = \frac{V_{2XX}}{2} + V_{3X} \tag{33}$$

$$V_{3Y} = \frac{1}{2} V_{3XX} + V_{4X} + V_2 V_{2X} \tag{34}$$

$$V_{2T} = \frac{1}{3} V_{2XXX} + V_{3XX} + V_{4X} + 2 V_2 V_{2X}. \tag{35}$$

Eliminating V_3 and V_4, we can obtain the KP equation $(V_2 = V)$

$$V_T = \tfrac{1}{4} V_{XXX} + V V_X + \partial_X^{-1} V_{YY}, \tag{36}$$

which also can be described as the compatibility condition for the eigenfunction ϕ

$$\phi_Y = \left(\tfrac{1}{2}\partial_X^2 + V\right)\phi \qquad \phi_T = \left(\tfrac{1}{3}\partial_X^3 + V\partial + V_3 + V_X\right)\phi. \tag{37}$$

To get dKP equation (5), one simply takes $T_n \to \varepsilon T_n = t_n$ in the KP equation (36) , with

$$\partial_{T_n} \to \varepsilon \partial_{t_n} \qquad \text{and} \qquad V(T_n) \to v(t_n),$$

to obtain the dKP equation when $\varepsilon \to 0$. Thus the dispersive term $\frac{1}{4} V_{XXX}$ is removed. Moreover, letting

$$\phi = \exp \frac{S}{\varepsilon}$$

in (37), we also have equation (6) for $n = 2, 3$

$$S_y = \tfrac{1}{2} S_x^2 + v_2 \qquad S_t = \tfrac{1}{3} S_x^3 + v_2 S_x + v_3$$

when $\varepsilon \to 0$. The compatibility $S_{ty} = S_{yt}$ will yield the dKP equation (5).

Since $v = p_1 - \tilde{p}_1$, from the theory of symmetry constraints of KP hierarchy [20, 21], one can assume the natural symmetry constraint

$$V = [1 - f(\partial_X)]^{-1} [\ln \phi_1 - \ln \phi_2]_X \tag{38}$$

where

$$f(\partial_X) = a_1 \partial_X + a_2 \partial_X^2 + \cdots + a_n \partial_X^n, \qquad a_i \text{ being constants.}$$

Here ϕ_1 and ϕ_2 are arbitrary eigenfunctions, i.e., they both satisfy equations (37). We remark that if $\phi_1 = \exp \frac{S_1}{\varepsilon}$, $\phi_2 = \exp \frac{S_2}{\varepsilon}$ and $X \to \varepsilon X = x$, then $V(X, Y, T) \to v(x, y, t) = p_1 - \tilde{p}_1$, where

$$p_1 = S_{1x}, \qquad \tilde{p}_1 = p_2 = S_{2x}$$

when $\varepsilon \to 0$.

Then equations (37) become, $i = 1, 2$,

$$\phi_{iY} = \frac{1}{2}\phi_{iXX} + \{[1 - f(\partial_X)]^{-1}(\ln\phi_1 - \ln\phi_2)_X\}\phi_i \tag{39}$$

$$\phi_{iT} = \frac{1}{3}\phi_{iXXX} + \{[1 - f(\partial_X)]^{-1}(\ln\phi_1 - \ln\phi_2)_X\}\phi_{iX} + [V_3 + \{[1 - f(\partial_X)]^{-1}(\ln\phi_1 - \ln\phi_2)_{XX}\}]\phi_i. \tag{40}$$

Then from (33) and (34) and (39), one can get

$$V_3 = \frac{1}{2}[1 - f(\partial_X)]^{-1}\{[(\ln\phi_1)_X]^2 - [(\ln\phi_2)_X]^2\}$$

$$V_4 = \frac{1}{3}[1 - f(\partial_X)]^{-1}\{[(\ln\phi_1)_X]^3 - [(\ln\phi_2)_X]^3 + 3[f(\partial_X)V^2] - 3V[f(\partial_X)V]\}.$$

Both the dispersionless limits of V_3 and V_4 are $v_3 = \frac{p_1^2 - \tilde{p}_1^2}{2}$ and $v_4 = \frac{p_1^3 - \tilde{p}_1^3}{3}$.

Now, using (35) and (40), a lengthy calculation shows that

$$\frac{[(\ln\phi_1)_{XX}]^2 - [(\ln\phi_2)_{XX}]^2}{2} = V[f(\partial_X)V_X] - f(\partial_X)(VV_X), \tag{41}$$

which is an contradiction since ϕ_1 and ϕ_2 are arbitrary eigenfunctions. This means that (40) is not a higher-order Lie–Backlund symmetry of (39) or the constraint (38) is not admissable [20, 35]. For example, letting $f(\partial_X) = 0$ and ∂_X, equation (41) becomes

$$(\ln\phi_1)_{XX}]^2 - [(\ln\phi_2)_{XX}]^2 = 0$$

and

$$[(\ln\phi_1)_{XX}]^2 - [(\ln\phi_2)_{XX}]^2 = -2V_X^2$$

respectively, both of which put constraints on ϕ_1 and ϕ_2 and are contradictions.

Finally, one notes that constraint (38) is not involved in adjoint eigenfunction and then it does not belong to the class considered in [21].

5. Concluding remarks

We have investigated the bi-Hamiltonian structure and dispersive corrections of the waterbag model for two components. After introducing suitable coordinates, one can identify (16) as a separable Hamiltonian system and thus a third-order bi-Hamiltonian structure is obtained. Also, using the recursion relation of conserved densities, we can find the free energy associated with waterbag model in WDVV equation of the topological field theory and thus establish the first-order bi-Hamiltonian structure. But the hierarchy constructed by the recursion operator is not the same as the waterbag hierarchy. Also, one considers the dispersive corrections from the theory of symmetric constraints of KP theory. Some calculations show that these dispersive corrections are not admissible. Finally, one remarks that the solutions of the waterbag model can be found using the hodograph method in [18, 19].

Several questions remain to be overcome. Firstly, from the theory of non-local Poisson brackets of hydrodynamic type [26, 28], one believes the bi-Hamiltonian structure of (29) (or (16)) deserves further investigations, especially that the free energy (27) is not quasi-homogeneous. Secondly, as we see in section 4, the integrable dispersive corrections of the waterbag model are still unknown. The main difficulty is in the quantization of the Lax operator (10), i.e., $p \to \partial_X$. The exact form is not clear and needs further investigations [36]. Ultimately, we hope to generalize the results in section 2 to the general case, for example, the four-component case. But the computation is more involved. One hopes to address these questions elsewhere.

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References

[1] Aoyama S and Kodama Y 1996 Topological Landau–Ginzburzg theory with a rational potential and the dKP hierarchy *Commun. Math. Phys.* **182** 185–220
[2] Arik M, Neyzi F, Nutku Y, Olver P J and Verosky J 1989 Multi-Hamiltonian structure of the Born–Infeld equation *J. Math. Phys.* **30** 1338–44
[3] Bogdanov L V and Konopelchenko B G 2004 Symmetry constrains for dispersionless integrable equations and systems of hydrodynamic type *Phys. Lett.* A **330** 448–59 (*Preprint* nlin.SI/0312013)
[4] Bogdanov L V and Konopelchenko B G 2004 On dispersionless BKP hierarchy and its reductions *Preprint* nlin.SI/0411046
[5] Boyarsky A, Marshakov A, Ruchayckiy O, Wiegmann P and Zabrodin A 2001 On associativity equation in dispersionless integrable hierarchies *Phys. Lett.* B **515** 483–92
[6] Carroll R and Kodama Y 1995 Solution of the dispersionless Hirota equations *J. Phys. A: Math. Gen.* **28** 6373–87
[7] Chang J-H and Tu M-H 2001 On the Benney hierarchy: free energy, string equation and quantization *J. Phys. A: Math. Gen.* **34** 251–72
[8] Cheng Yi 1992 Constraints of the Kadomtsev–Petviashvili hierarchy *J. Math. Phys.* **33** 3774–82
[9] Dickey L A 2003 *Soliton Equations and Hamiltonian Systems* 2nd edn (Singapore: World Scientific)
[10] Dubrovin B and Novikov S P 1989 Hydrodynamics of weakly deformed soliton lattices, differential geometry and Hamiltonian theory *Russ. Math. Surv.* **44** 35–124
[11] Dubrovin B A and Novikov S P 1983 Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov–Whitham averaging method *Sov. Math. Dokl.* **27** 665–9
[12] Dubrovin B 1996 Geometry of 2D topological field theories *Integrable Systems and Quantum Groups* (*Springer Lecture Notes in Math.* vol 1620) (Berlin: Springer) pp 120–348
[13] Dubrovin B 1998 Flat pencils of metrics and Frobenius manifolds *Preprint* math.DG/9803106
[14] Dubrovin B and Zhang Y J 2001 Normal forms of hierarchies of integrable PDES, Frobenius manifolds and Gromov–Witten invariants *Preprint* math.DG/0108160
[15] Dubrovin B 1992 Integrable systems in topological field theory *Nucl. Phys.* B **379** 627–89
[16] Gibbons J and Tsarev S P 1996 Reductions of the Benney equations *Phys. Lett.* A **211** 19–24
[17] Gibbons J and Tsarev S P 1999 Conformal maps and reductions of the Benney equations *Phys. Lett.* A **258** 263–71
[18] Kodama Y and Gibbons J 1990 Integrability of dispersionless KP hierarchy *Proc. Fourth Workshop on Nonlinear and Turbulent Process in Physics* (Singapore: World Scientific) pp 166–80
[19] Kodoma Y 1988 A solution method for dKP equation *Prog. Theor. Phys. Suppl.* **94** 184–94
[20] Konopelchenko B and Strampp W 1992 Reductions of (2+1)-dimensional integrable systems via mixed potential-eigenfunction constraints *J. Phys. A: Math. Gen.* **25** 4399–411
[21] Konopelchenko B and Strampp W 1992 New reductions of Kadomtsev–Petviashvili and two-dimensional Toda lattice hierarchies via symmetry constraints *J. Math. Phys.* **33** 3676–86
[22] Krichever I 1994 The τ-function of the universal Whitham hierarchy, matrix models and topological field theories *Commun. Pure Appl. Math.* **47** 437–75
[23] Magri F 1978 A simple model of the integrable Hamiltonian equation *J. Math. Phys.* **19** 1156–62
[24] Maltsev A Ya 2002 On the compatible weakly non-local Poisson brackets of hydrodynamic type *Int. J. Math. Math. Sci.* **32** 587–614
[25] Maltsev A Ya and Novikov S P 2001 On the local systems Hamiltonian in the weakly non-local Poisson brackets *Phys.* D **156** 53–80
[26] Mokhov I O 2002 Compatible nonlocal Poisson brackets of hydrodynamic type, and integrable hierarchies related to them *Preprint* math.DG/0201242
[27] Mokhov I O 2003 Quasi-Frobenius algebras and their integrable N-parametric deformations generated by compatible $(N \times N)$-metrics of constant Riemannian curvature *Theor. Math. Phys.* **136** 908–16

[28] Mokhov O I 2004 Nonlocal Hamiltonian operators of hydrodynamic type with flat metrics, integrable hierarchies and the equations of associativity *Preprint* math.DG/0406292

[29] Li L-C 1999 Classical r-matrix and compatible Poisson structures for Lax equations on Poisson algebras *Commun. Math. Phys.* **203** 573–92

[30] Nutku Y and Pavlov M V 2002 Multi-Lagrangians for integrable systems *J. Math. Phys.* **43** 1441–59

[31] OLver P J 1993 *Applications of Lie-Groups to Differential Equations* (New York: Springer)

[32] Olver P J and Nutku Y 1988 Hamiltonian structures for systems of hyperbolic conservation laws *J. Math. Phys* **29** 1610–9

[33] Pavlov M V 2004 Classifying integrable hydrodynamic Egoroff chains *Theo. Math. Phys.* **138** 45–58

[34] Sheftel M B 2001 Group analysis of hydrodynamic-type systems *Preprint* math-th/0104014

[35] Sidorenko J and Strampp W 1991 Symmetry constraints of the KP hierarchy *Inverse Problems* **7** 37–43

[36] Strachan I A B 1997 A geometry for multidimensional integrable systems *J. Geom. Phys.* **21** 255–78

[37] Takasaki K and Takebe T 1995 Integrable hierarchies and dispersionless limit *Rev. Math. Phys.* **7** 743

[38] Zakharov V E 1980 Benney equation and quasi-classical approximation in the inverse problem method *Funkts. Anal. Prilozh* **14** 89–98